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O spojitých a nikde diferencovatelných funkcích

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Katedra matematiky, FAV ZČU v Plzni

Občasný seminář z matematické analýzy (OSMA)

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evropský
sociální
fond v ČR



EVROPSKÁ UNIE



MINISTERSTVO ŠKOLSTVÍ,
MLÁDEŽE A TĚLOVÝCHOVY



OP Vzdělávání
pro konkurenceschopnost

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Definice 1.

Mějme posloupnost funkcí (f_n) , které jsou definovány na $D \subset \mathbb{R}$.

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- Řekneme, že (f_n) **konverguje bodově** na $M \subset D$ k funkci f , pokud

$$\forall x \in M : \lim_{n \rightarrow +\infty} f_n(x) = f(x),$$

tj. pokud

$$\forall x \in M \quad \forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \in \mathbb{N} : n > n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

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- Řekneme, že (f_n) **konverguje stejnoměrně** na $M \subset D$ k funkci f , pokud

$$\lim_{n \rightarrow +\infty} \left(\sup_{x \in M} |f_n(x) - f(x)| \right) = 0,$$

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Věta 2.

Mějme interval $I \subset \mathbb{R}$.

$$\left. \begin{array}{l} \text{1 } \forall n \in \mathbb{N} : f_n \text{ je spojitá na } I, \\ \text{2 } f_n \Rightarrow f \text{ na } I, \end{array} \right\} \Rightarrow f \text{ je spojitá na } I.$$

Věta 2.

Mějme interval $I \subset \mathbb{R}$.

$$\left. \begin{array}{l} \text{1 } \forall n \in \mathbb{N} : f_n \text{ má vlastnost } V \text{ na } I, \\ \text{2 } f_n \Rightarrow f \text{ na } I, \end{array} \right\} \Rightarrow f \text{ má vlastnost } V \text{ na } I.$$

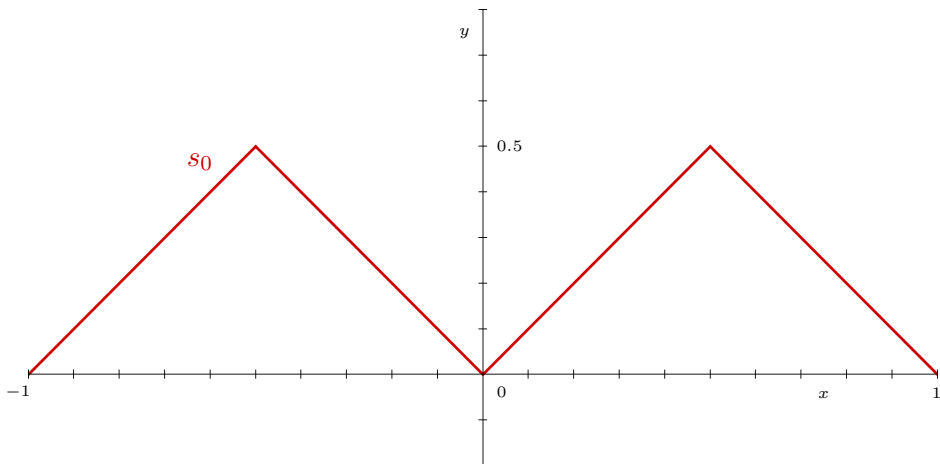
Quiz: Pro kterou vlastnost V bude předchozí **Věta** platit?

- ▶ spojitost,
- ▶ omezenost,
- ▶ sudost,
- ▶ lichost,
- ▶ Riemannovská integrovatelnost,
- ▶ diferencovatelnost,
- ▶ lipschitzovskost.

Konstrukce Takagiho funkce

$$s_0(x) = \varphi(x)$$

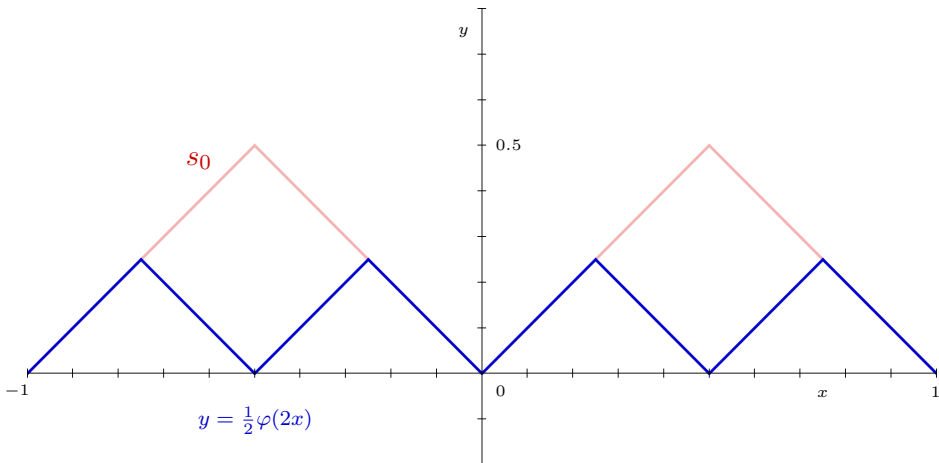
$$\varphi(x) = \text{dist}(x, \mathbb{Z}) = \inf_{m \in \mathbb{Z}} |x - m|.$$



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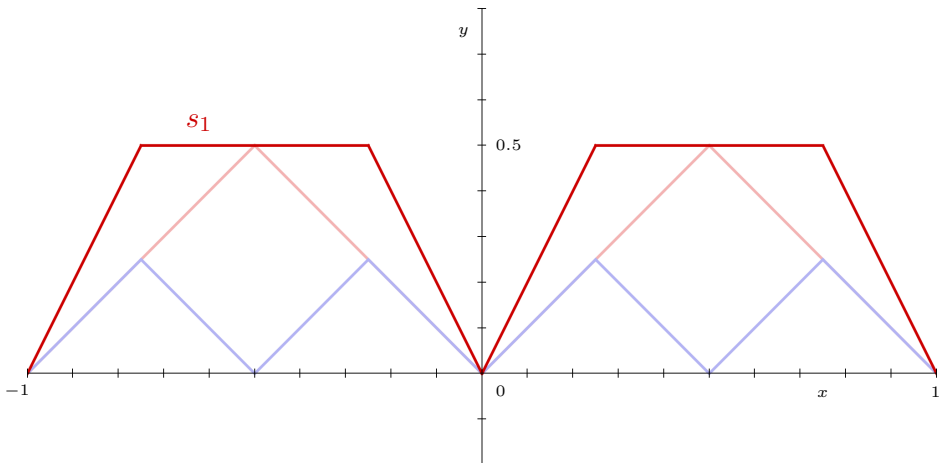
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Konstrukce Takagiho funkce

$$s_1(x) = \varphi(x) + \frac{1}{2}\varphi(2x)$$

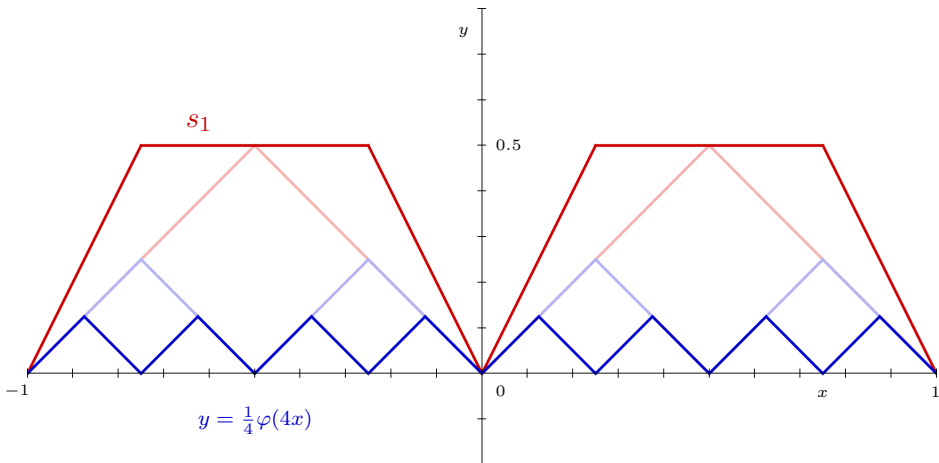
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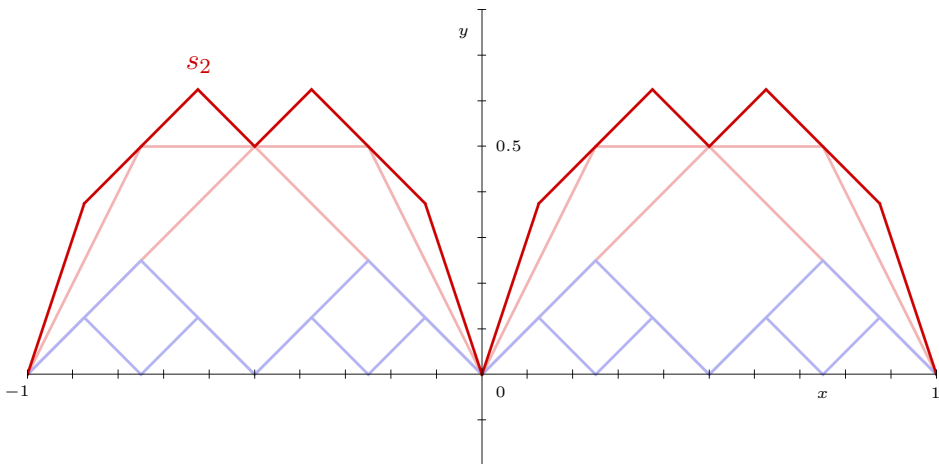
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$$s_2(x) = \varphi(x) + \frac{1}{2}\varphi(2x) + \frac{1}{4}\varphi(4x)$$

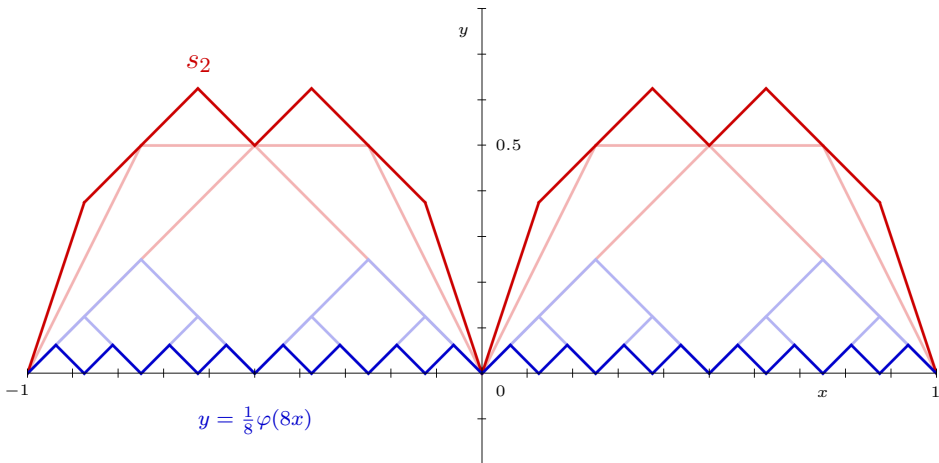
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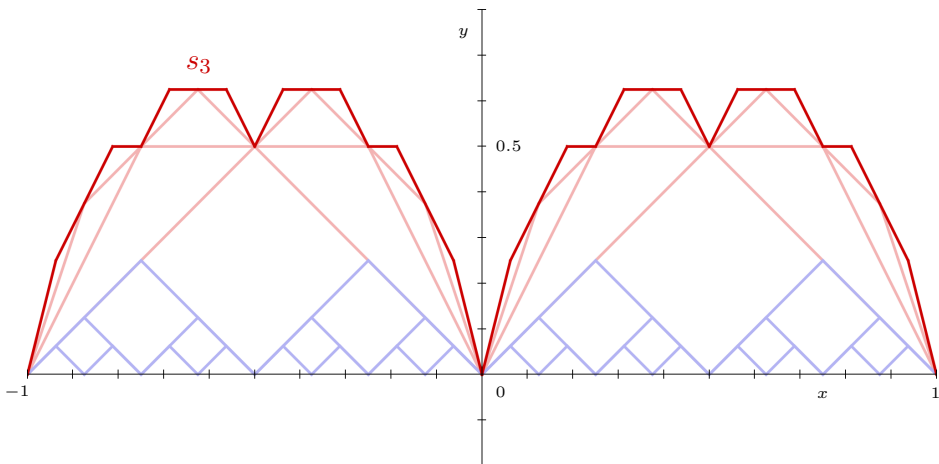
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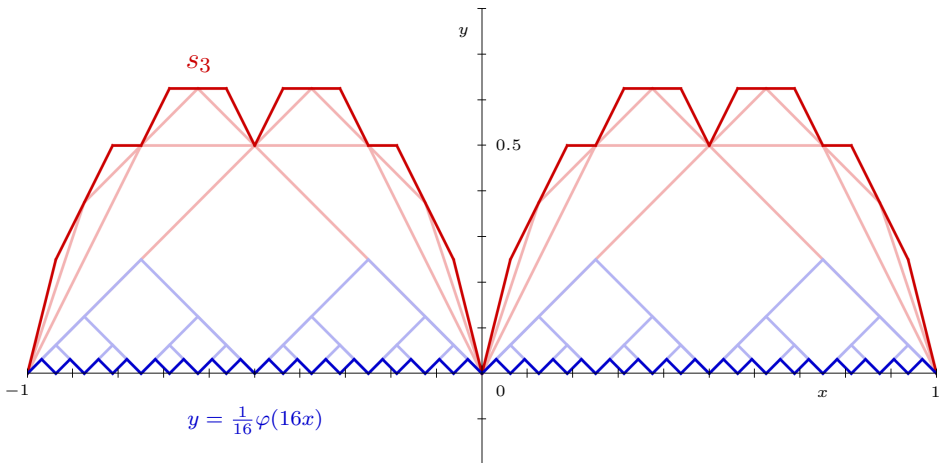
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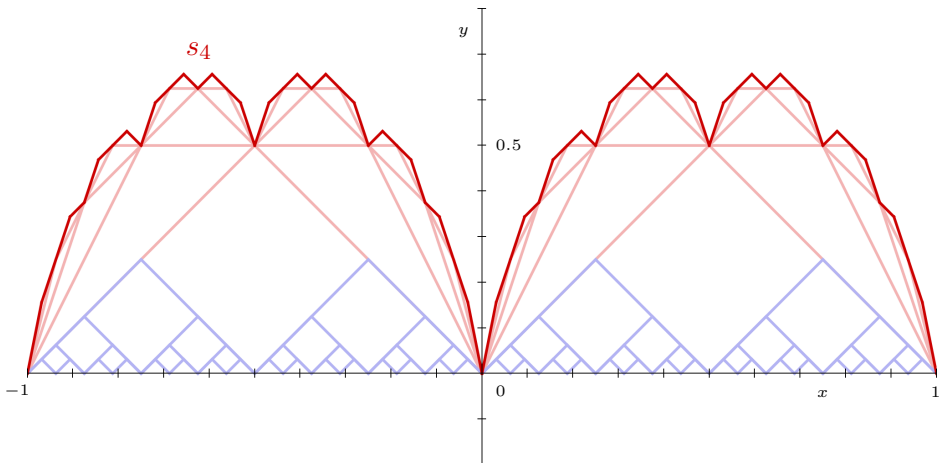
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Konstrukce Takagiho funkce

$$s_4(x) = \varphi(x) + \frac{1}{2}\varphi(2x) + \frac{1}{4}\varphi(4x) + \frac{1}{8}\varphi(8x) + \frac{1}{16}\varphi(16x)$$

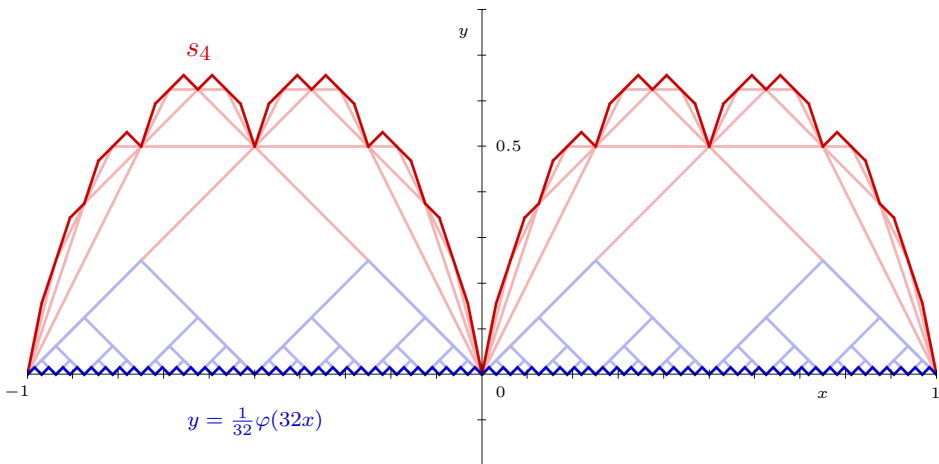
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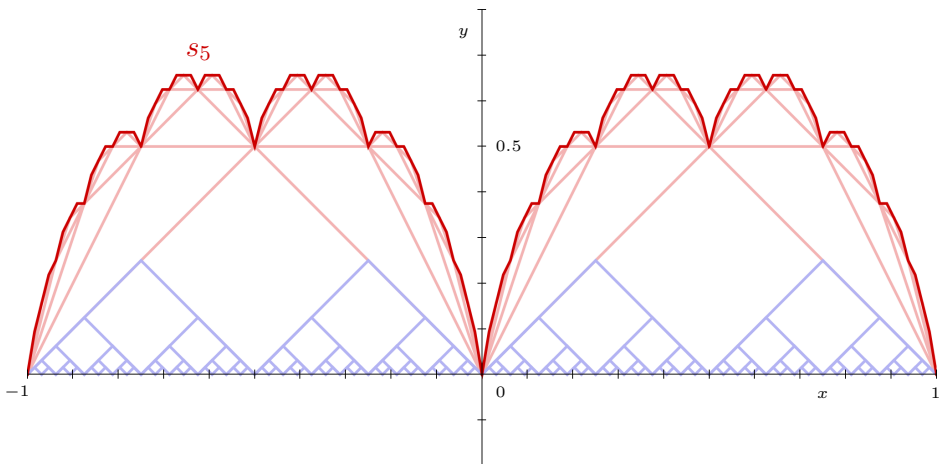
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Konstrukce Takagiho funkce

$$s_5(x) = \varphi(x) + \frac{1}{2}\varphi(2x) + \frac{1}{4}\varphi(4x) + \frac{1}{8}\varphi(8x) + \frac{1}{16}\varphi(16x) + \frac{1}{32}\varphi(32x)$$

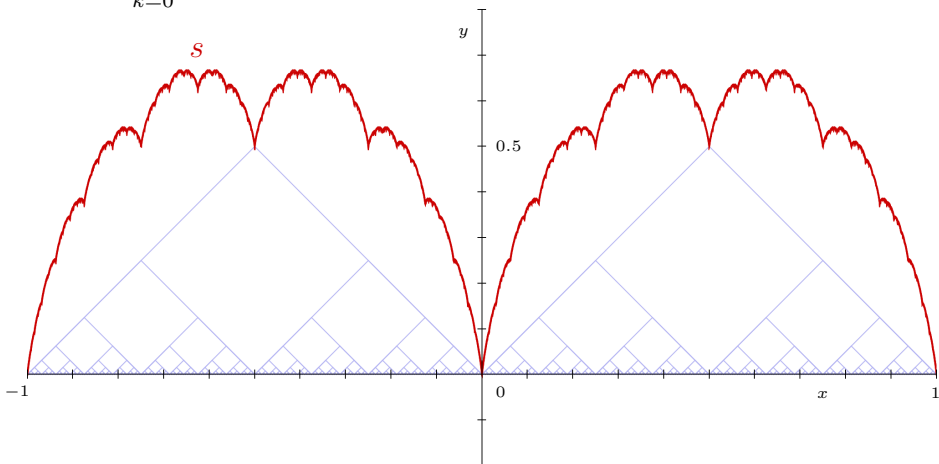
$$\varphi(x) = \text{dist}(x, \mathbb{Z}) = \inf_{m \in \mathbb{Z}} |x - m|.$$



Konstrukce Takagiho funkce

$$\begin{aligned}
 s(x) &= \varphi(x) + \frac{1}{2}\varphi(2x) + \frac{1}{4}\varphi(4x) + \frac{1}{8}\varphi(8x) + \frac{1}{16}\varphi(16x) + \frac{1}{32}\varphi(32x) + \dots \\
 &= \sum_{k=0}^{+\infty} \frac{1}{2^k} \varphi(2^k x),
 \end{aligned}$$

$$\varphi(x) = \text{dist}(x, \mathbb{Z}) = \inf_{m \in \mathbb{Z}} |x - m|.$$



Věta 3. (Bolzanova-Cauchyova podmínka)

(s_n) konverguje stejnoměrně na M



$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall m, n \in \mathbb{N} : \quad m, n > n_0 \quad \Rightarrow \quad \sup_{x \in M} |s_m(x) - s_n(x)| < \varepsilon$$

$$\lim_{m, n \rightarrow +\infty} \left(\sup_{x \in M} |s_m(x) - s_n(x)| \right) = 0$$

Věta 4. (Weierstrassovo kritérium)

Mějme řadu funkcí $f_1(x) + f_2(x) + f_3(x) + \cdots = \sum_{n=1}^{+\infty} f_n(x)$.

Nechť $b_1 + b_2 + b_3 + \cdots = \sum_{n=1}^{+\infty} b_n$ je číselná řada taková, že

$$1 \quad \forall n \in \mathbb{N} \quad \forall x \in M : |f_n(x)| \leq b_n,$$

$$2 \quad \sum_{n=1}^{+\infty} b_n \text{ konverguje.}$$

Potom řada funkcí $\sum_{n=1}^{+\infty} f_n(x)$ **konverguje stejnoměrně** na M .

Věta 5.

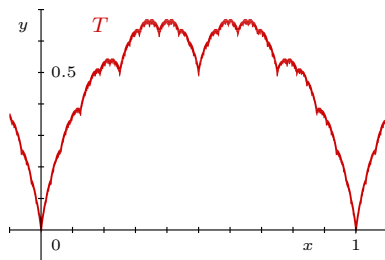
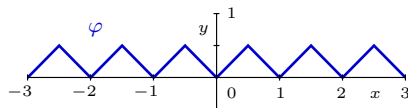
Takagiho funkce

$$T(x) = \sum_{k=0}^{+\infty} \frac{1}{2^k} \varphi(2^k x),$$

kde

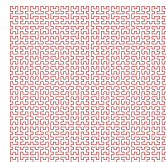
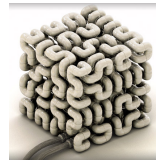
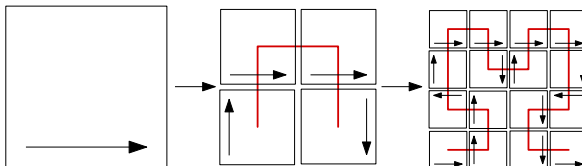
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je spojitá a nikde diferencovatelná funkce na $\mathbb{R}$.

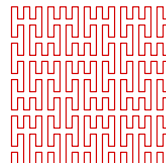
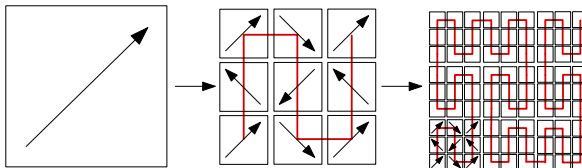


Plochu-vyplňující křivky

► Hilbertova křivka



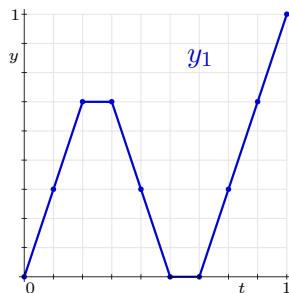
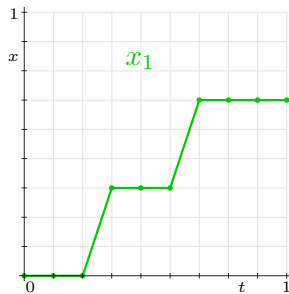
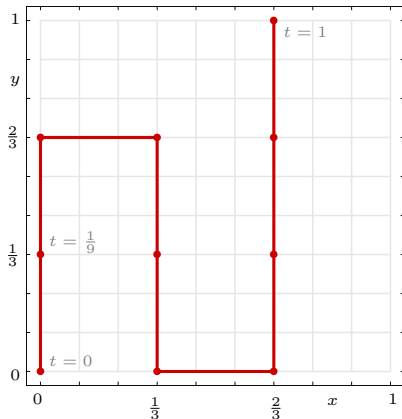
► Peanova křivka



Konstrukce Peanovy křivky

$$\gamma_n: \langle 0, 1 \rangle \rightarrow \langle 0, 1 \rangle \times \langle 0, 1 \rangle$$

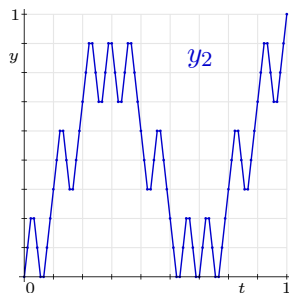
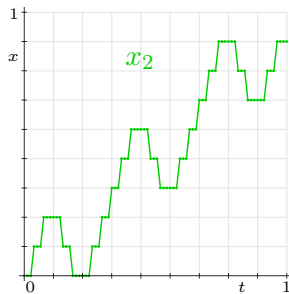
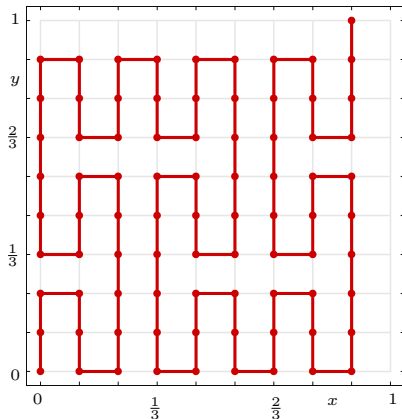
$$\gamma_n: t \mapsto (x_n(t), y_n(t))$$

 γ_1


Konstrukce Peanovy křivky

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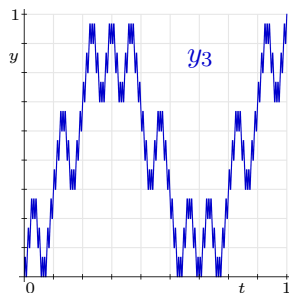
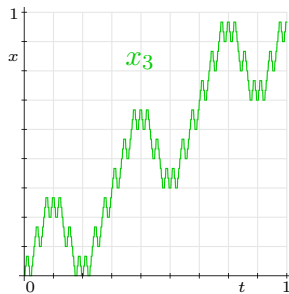
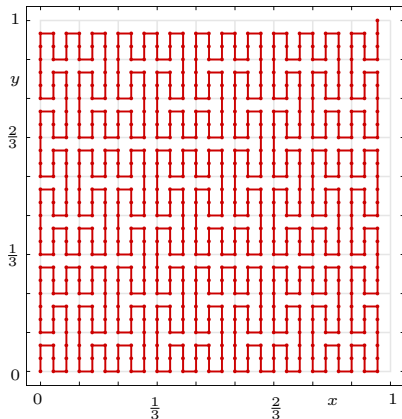
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 γ_2


Konstrukce Peanovy křivky

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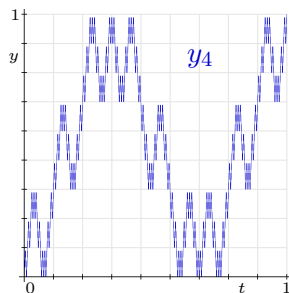
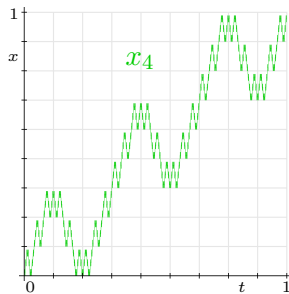
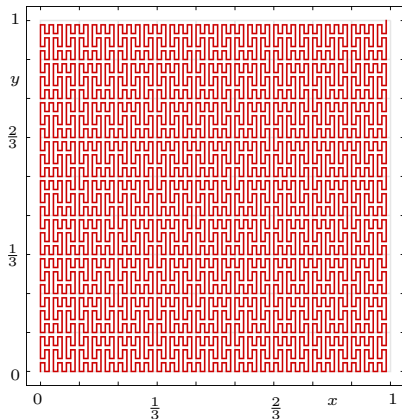
$$\gamma_n : t \mapsto (x_n(t), y_n(t))$$

 γ_3


Konstrukce Peanovy křivky

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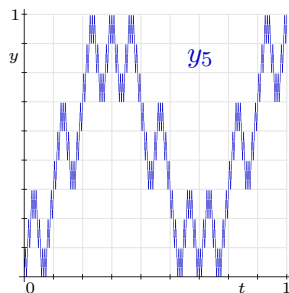
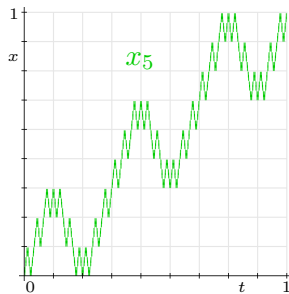
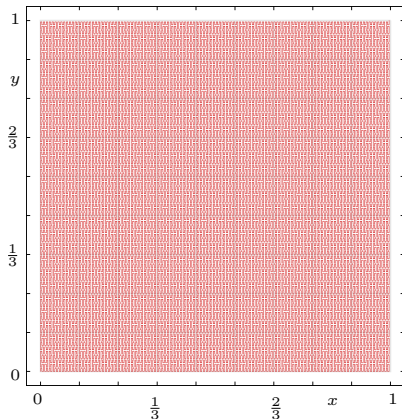
 γ_4


Konstrukce Peanovy křivky

$$\gamma_n : \langle 0, 1 \rangle \rightarrow \langle 0, 1 \rangle \times \langle 0, 1 \rangle$$

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γ_5

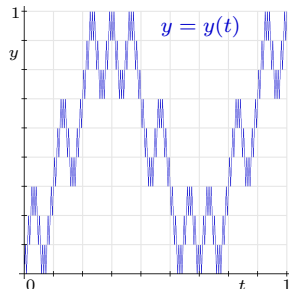
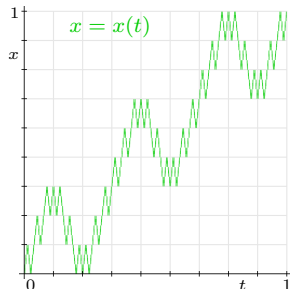


Věta 6.

Obě složky $x = x(t)$ a $y = y(t)$ Peanovy křivky

$$\left\{ \begin{array}{l} \gamma: \langle 0, 1 \rangle \rightarrow \langle 0, 1 \rangle \times \langle 0, 1 \rangle, \\ \gamma: (t_0 \ t_1 t_2 \dots)_3 \mapsto \left(\begin{array}{l} (0 \ (k^{t_0} t_1) (k^{t_0+t_2} t_3) (k^{t_0+t_2+t_4} t_5) \dots)_3 \\ (t_0 \ (k^{t_1} t_2) (k^{t_1+t_3} t_4) (k^{t_1+t_3+t_5} t_6) \dots)_3 \end{array} \right), \end{array} \right.$$

jsou spojité a nikde diferencovatelné funkce na intervalu $\langle 0, 1 \rangle$.



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